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NONLINEAR ANALYSIS OF PHYSICAL LIBRATIONS

I. New Formulation and Equilibrium Orientations Deduced From an Integral of Motion

by R. L. Duncombe and I. Michelson

Prepared under Grant No. NsG-561 *by*

ILLINOIS INSTITUTE OF TECHNOLOGY
Chicago, Ill.

for

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Introduction

Satellite orientation in space is affected by the unequal attractions of its parts toward the distant centroid of its primary, which is the Earth both for artificial satellites and for the Moon. In passive systems of satellite attitude control, the resultant torque attributable to the gradient of gravity forces within the satellite is of dominant importance. Both Lagrange and Laplace gave the theoretical basis for discussion of satellite rotations in their analyses of lunar librations (Ref. 1, 2). Hamiltonian methods and modern computer techniques permit more accurate calculations for satellites of arbitrary configuration in generalized orbits. With respect to lunar librations also, new insights are provided on persistent difficulties in the classical treatment.

author

For more than a hundred years careful measurements have been made, chiefly with equipment specially designed by Bessel, in order to evaluate the gross dynamic features of lunar structure according to the theory proposed by Laplace. The differences between principal moments of inertia of the Moon are represented by a single parameter which has been sought in this manner, but which has not yet been well established. It is largely for this reason that Commission 17 of the International Astronomical

* Presented at XVth International Astronautical Congress, Warsaw, September 11, 1964.

Union unanimously affirmed in 1948 the importance of the introduction of a rigorous method for the reduction of all accumulated observations of physical lunar libration. Noting that this has not yet been achieved, both Sh.T. Khabibullin and K. Koziel have recently endorsed that resolution (Ref. 3, 4). Although virtually all of the effort which has been devoted to the study of lunar librations since the time of Laplace has rested upon the analysis which he gave, the state of affairs just described is here taken to justify a critical review and revision of that time-honored theory.

Before considering a reformulation of that work, therefore, it is appropriate to indicate some basic features of Laplace's calculation which deserve to be ameliorated. Laplace recognized that the very small amplitudes of librational motion provide a valid basis of approximation in the neglect of powers and products of small quantities. In addition to these approximations which lead to a system of linear differential equations in a manner that is now familiar, Laplace also applied the same technique with regard to quantities that define the lunar orbit and lunar orientation in space. These include, for example, lunar orbit eccentricity (value 0.055), and the angle between lunar equatorial and orbital planes (value $6° 44'$), values not in fact so small as to justify the assumptions when an accurate theory is sought which is found to be sensitive to the values assigned to physical parameters. Other parameters now known with better precision and different from values available to Laplace, are likewise important to recognize; notable among these is the inclination of the lunar equator, as given by C.B. Watts in 1955, Ref. 5. The introduction of Hamiltonian methods in dynamics within the past century, together with the associated perturbation techniques, will also be seen to have far-reaching significance in the present problem; viewed by current standards, their omission in the past is recognized as a serious defect of the theory. Not unrelated to this is a prevalent ambiguity concerning the appropriate form of a principle of angular momentum, obscuring essential features of the motion; this point will be elaborated below. Finally it will be seen that the classical theory contains logical imperfections which hinder solution of the problem to determine the physical constants of libration, although these were of secondary importance in the calculations which Laplace made for the purpose of testing hypotheses on the primitive state of the Moon.

With regard to artificial satellites of general configuration in elliptic orbits, moreover, it is clear that the special assumptions made by Laplace are entirely inappropriate, so that a fresh formulation of the dynamical theory is indispensible for an acceptable description of the motion. In the contemporary literature on satellite librations, it should be noted that the viewpoint adopted by V.V. Beletskii (Ref. 6), G.N. Duboshin (Ref. 7), D.E. Okhotsimski (Ref. 8) and their co-workers is that of the more general nonlinear treatment of satellite motion, configurations and orbits being unrestricted. An essential difference of their work from the present analysis is their use of Euler's equations of rigid body motion without modification for relative motion; this feature will be analyzed below.

Reference Frames and Angular Coordinates

It is customary in discussions of rotational motions of Earth and Moon, to employ a set of reference axes with origin located at the centroid of the body in question and directions parallel to the three principal inertia axes. Then products of inertia are all zero, and the principle of angular momentum is expressed by the set of three differential equations known by the name of Euler. Relative motion between bodies is accounted as an external forcing, the representation being that suggested by the conception of celestial sphere, long familiar and useful in astronomy. In this fashion it is possible to discuss, for example, either the rotational motion of Earth as influenced by the Moon, or that of Moon under influence of Earth, the formalism being the same in both cases. Such was, in fact, the order of development in Laplace's "Mécanique Célèste" and subsequent treatises respect the tradition without important exception.

In modern times there is no lack of awareness of the pitfalls encountered when the classical laws of mechanics are not expressed relative to an inertial reference frame. Hence even for the discussion of Earth rotation as affected only by Moon and Sun, it is realized that reference axes in motion with Earth centroid cannot be treated strictly as an inertial system -- centroid of Earth-Moon system is considerably removed from that of Earth alone, and the heliocentric motion itself must sometimes be reckoned also. Still less accurately can Moon's rotational motion be described as if its centroid were fixed relative to an inertial reference frame. The concept of gravitating centers has been employed by M. Nahas, to account for effects of non-inertial reference frame in dealing with tidal theory (Ref. 9). These niceties are of no physical importance when effects of relative motion are adequately accountable in terms of external forces, but it is by no means clear that this has been shown in the analysis of lunar librations -- nor even that this procedure has the possibility of being sufficient in this regard. Laplace's discussion does not mention these questions at all.

For the purposes of careful calculation, therefore, it is convenient that the principle of least action (Hamilton's principle) is established without reference to coordinate systems. The appropriate form of the principle of angular momentum in an arbitrarily moving reference frame is therefore assured when the relevant energies are expressed in terms of the coordinates of that reference frame. It will be seen in the next section that the Euler equations require modification when satellite librational motion is considered. Since significant consequences follow this fact, a detailed discussion and derivation are indicated.

Of less fundamental interest is the use of an angle convention different from the traditional representation, motivated by the following considerations. In the Hamiltonian formulation which is adopted, there is the inherent advantage of having no requirement for limitation to small angular displacements, obviating the necessity for early recourse to linearization approximations. Both as a means of assessing such approximations when and if they are later imposed (as they are in the traditional treatments of lunar librations) and also in the interest of generality for discussion of large amplitude motions of other satellites, the differential equations of motion are written in the more complete and nonlinear form. Euler's angle convention is of course well suited to this purpose. But it is also desirable to have the possibility to discuss arbitrary small rotations, and here the Euler angles are less convenient. The reason is seen when it is recalled that the ordered sequence of rotations in the Euler convention entails a rotation about one axis, followed by a second rotation about a perpendicular axis, and a final rotation about the first axis. When these angles are all small, the resultant rotation about that axis which is perpendicular to both of the foregoing, is small to a higher order than the two angular displacements themselves. In order to represent a less restrictive resultant rotation, the angles being small, a more symmetric angle convention better serves the purpose. In the convention adopted in the present discussion, this is accomplished by an ordered set of three rotations, one about each of three mutually orthogonal axes. (Cf., e.g., Ref. 14).

Specifically, axes X_1, X_2, X_3 are located with origin at satellite (hereafter simply called "Moon") centroid, such that X_1, is normal to orbit plane in direct sense; X_3 along the line of Earth and Moon centroids, directed away from Earth; X_2 completes a right-handed system, so that X_2 would be directed opposite to orbital velocity if the orbit were circular. The ordered sequence of rotations is given by angle α in the positive sense around axis X_1, followed by β in the positive sense around the once-displaced axis X_2, and finally by γ in the positive sense around the twice-displaced axis X_3. The composite rotation locates principal inertia axes of Moon, denoted by x_1, x_2, x_3, respectively. The rotations are illustrated in Fig. 1, where the direction cosines for the coordinate transformations are also shown in matrix form along with explicit values of the direction cosines as functions of α, β, γ.

Earth-Moon centroid distance is denoted by $\bar{R}$, the direction of the vector being taken from Earth to Moon, its magnitude being represented by R. The orbital angular velocity $\bar{\Omega}$ is parallel to the axis X_1, its magnitude Ω (as well as R) being dependent on position in orbit (assumed planar and Keplerian) as given by the true anomaly. Distance from lunar centroid to an arbitrary mass point of the Moon is given by $\bar{r}$, having cartesian components referred to principal axes given by x_1, x_2, x_3. Principal inertia moments about these axes are respectively denoted by A, B, C; these are subject to the inequalities

$$A > B > C$$

for stable orientations corresponding to rotation angles α and β each in some neighborhood of zero. Angular velocity components of librational motion are denoted p, q, r, their vector resultant $\bar{\omega}$. It is readily shown that these are expressed in terms of the angle-rates $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, the direction cosines and the angle γ by the formulas

$$\begin{aligned} p &= a_1\dot{\alpha} + \sin\gamma\,\dot{\beta} \\ q &= b_1\dot{\alpha} + \cos\gamma\,\dot{\beta} \\ r &= c_1\dot{\alpha} + \dot{\gamma} \quad . \end{aligned} \tag{1}$$

Lagrangian Function, Orbit Eccentricity Neglected

It is useful to establish important features of the librational motion directly from the Lagrangian function, the required expressions being simplified by neglecting orbit eccentricity, although analogous deductions remain valid when this assumption is later relaxed. For further simplicity in discussion of lunar rotation, Earth-centroid is treated as executing a motion which qualifies it as appropriate location of an inertial reference frame. In this manner the non-inertial character of Moon-fixed axes is partially accounted and will be seen to introduce centrifugal couples due to the lunar motion while ignoring those caused by the annual motion of

Earth (the neglected terms can be shown to be dwarfed by the terms which are retained).

The velocity of a point distant $\bar{r}$ from lunar centroid is then given by

$$\bar{v} = \bar{v}_o + (\bar{\Omega} + \bar{\omega}) \times \bar{r} \tag{2}$$

where $\bar{v}_o$ is the linear velocity of lunar centroid in its orbit, given as the outer vector product in the expression

$$\bar{v}_o = \bar{\Omega} \times \bar{R}.^{*}$$

It is clear from (2) that the absence of librational motion, corresponding to $\bar{\omega} = 0$, is an orbital motion with rotation $\bar{\Omega}$. Whereas orbital motion without rotation corresponds to

$$\bar{\Omega} + \bar{\omega} = 0,$$

and is altogether physically possible, the present decomposition of total rotational motion is more convenient since librational motion can be immediately identified as $\bar{\omega}$. It is to be recalled that $\bar{\Omega}$ is in the direction X_1, while the components p, q, r of the vector $\bar{\omega}$ are parallel to x_1, x_2, x_3, respectively.

The orbital kinetic energy being constant for circular orbit, as well as Ω and R, the part of the kinetic energy which affects the librational motion is contained entirely in the integral over all mass elements dm, of the scalar product of the rotational motion with itself:

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* The magnitude of $\bar{r}$ is not required and will not be employed; hence the conventional symbol r for one component of librational motion entails no ambiguity.

$$T = 1/2\int\{(\bar{\Omega} + \bar{\omega}) \times \bar{r}\}\cdot\{(\Omega + \bar{\omega}) \times \bar{r}\}\, dm \quad .$$

In integrated form this is

$$T = \frac{A}{2}(a_1\Omega + p)^2 + \frac{B}{2}(b_1\Omega + q)^2 + \frac{C}{2}(c_1\Omega + r)^2 \qquad (3)$$

$$\text{where } a_1 = \cos\beta\cos\gamma$$

$$b_1 = -\cos\beta\sin\gamma$$

$$c_1 = \sin\beta$$

and p, q, r are given by (1). Interpretation of each term in (3) is immediate: the terms involving Ω represent orbital centrifugal energy, p, q, r that due to librational motion, and the two separate limiting cases each yield values that are recognizable. The orbital centrifugal terms represent the relative motion of Earth and Moon and are seen in (3) to entail, in addition to terms proportional to Ω^2 and independent of librational motion p, q, r, also terms such as $Aa_1\Omega p$. Hence although p, q, r are homogeneous functions of degree unity in the angle rates $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, it is clear that the kinetic energy is not a homogeneous function of these variables, and also that centrifugal effects cannot be entirely separated from librational motion. This separation is attempted in the classical treatment of librations, the energy of motion being written without the Ω-terms (see, e.g., Tisserand, Ref. 10, p. 374), all effects of relative motion being referred to a potential, to be examined in detail below. The kinetic energy referred to axes moving with Moon centroid is given by terms proportional to p^2, q^2, r^2 alone; the transformation of energies indicated by (3) for an Earth-centered reference frame is in accord with textbook demonstrations (see, e.g., Landau & Lifshitz, Ref. 11, p. 18).* The subsequent discussion will show what consequences follow the retention of centrifugal terms in (3). At this point it is noted that the kinetic

* The two terms in equation (7) which are due to the relative motion are also found in the analysis of tidal and other dynamical theories; a close parallel is found in Ref. 15.

energy, containing as it does terms proportional to the first power of angle rates $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, may be regarded as similar to certain gyroscopic cases; systems with energies of this form are also sometimes terms "unnatural".

Recognizing the correctness in principle of the centrifugal terms in (3), their importance in particular cases depends on how large the dynamic consequences are, compared with other terms. For rotational motions at considerable speed, the components of $\bar{\omega}$ are so much greater than $\bar{\Omega}$ that the latter may be neglected -- this is evidently the case for gyroscopes and a wide class of other motions. The distinctive feature of librational motions, on the other hand, is the smallness of these terms. The lunar librations, in fact, are closely described by the so-called Cassini's Three Laws, according to which these effects are exactly zero; theoretical studies of the lunar librations are concerned with precisely the actual departures from zero of the velocities p, q, r. Even in the limiting case $p = q = r = 0$, the importance of the Ω terms depends on the magnitude of the "external" forces which, if much greater, will determine the librational configuration almost without regard to the centrifugal terms. It will be seen at once, however, that when the principal forces are those due to Earth attraction, as in the case of lunar librations, the two sets of terms are of exactly the same order of magnitude; then Cassini's laws describe the equilibrium established by the balance of gravitational and centrifugal couples. In any case it is useful to observe from (3) that the kinetic energy is a function of the angle rates $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ and also, the angles themselves, which determine the direction cosines a_1, b_1, c_1.

The potential energy may be written to a satisfactory degree of approximation as the sum of a term inversely proportional to the Earth-Moon distance R, plus another involving the third power of the same distance and the moment of inertia differences. This gives

$$U = \frac{-K}{R} - \frac{K}{2R^3}(A + B + C - 3I)$$

where K denotes the product of Earth mass by the constant of gravitational attraction and I is the moment of inertia of the Moon about the Earth-Moon line of centroids. According to a well-known formula this gives

$$I = Aa_3^2 + Bb_3^2 + Cc_3^2$$

where a_3, b_3, c_3 are indicated in Fig. 1 and have the values

$$a_3 = \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma$$

$$b_3 = \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma$$

$$c_3 = \cos\alpha\cos\beta \quad .$$

Then the parts of U which depend on α, β, γ are

$$\frac{-K}{2R^3}\{A(1-3a_3^2) + B(1-3b_3^2) + C(1-3c_3^2)\} \quad .$$

Noting that the force balance in circular orbit gives

$$\Omega^2 = \frac{K}{R^3}$$

and regrouping terms, the "tidal" potential is expressed in terms of the asymmetries of lunar mass distribution by the formula

$$U = \frac{-\Omega^2}{2}\{(A-B)(3b_3^2-1) + (A-C)(3c_3^2-1)\} \quad . \tag{4}$$

Equation (4), sometimes referred to as MacCullagh's formula, may be compared with the form given, for example, on p. 378 of Reference 10. It is to be noted that the energy of lunar orientation, expressed by (4), is a function of the angles α, β, γ; evidently it vanishes in the case of spherical (or other) symmetry for which A = B = C.

The Lagrangian function L is given by the difference T-U given by (3) and (4), and this is a function of the angles α, β, γ and their derivatives, but it is not explicitly dependent on time.

The Integral of Energy and Permanent Configurations

The circumstance that the Lagrangian function does not depend explicitly on the time simplifies the expression of the principle of least action. In the present case, having three coordinates α, β, γ, the usual system of three second-order differential equations ("Lagrange's Equations") is replaced by the single condition

$$\dot{\alpha}\frac{\partial L}{\partial \dot{\alpha}} + \dot{\beta}\frac{\partial L}{\partial \dot{\beta}} + \dot{\gamma}\frac{\partial L}{\partial \dot{\gamma}} - L = \text{const.} \tag{5}$$

the derivative of which with respect to time is seen to give

$$\dot{\alpha}\{\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} - \frac{\partial L}{\partial \alpha}\} + \dot{\beta}\{\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} - \frac{\partial L}{\partial \beta}\} + \dot{\gamma}\{\frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}} - \frac{\partial L}{\partial \gamma}\} = 0 \tag{6}$$

and therefore yield the following interpretation. In the event that $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ have an arbitrary set of values at any instant, the coefficients of each of these terms in (6) must vanish separately. These are Lagrange's Equations, and in the particular case of rigid body motion, give the familiar set of equations associated with the name Euler. But if $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$, all at the same time, then equation (6) is equally satisfied without equating to zero each of the coefficients in brackets, i.e., without recourse to Lagrange's equations in general, or to the Euler rigid-body equations in the case of librational motion.

Noting that Cassini's laws are very nearly true, i.e., that $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$ to a good approximation, it is interesting to examine (5) in this case, since it then represents the complete dynamical requirement according to Hamilton's Principle. At this point the presence of centrifugal terms in (3) leads to unfamiliar consequences which are examined from the explicit form taken by (5) after assigning zero values to $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$. This gives

$$(A-B)(3b_3^2-b_1^2) + (A-C)(3c_3^2-c_1^2) = \text{const.} \tag{7}$$

when additive constant B+C-A and multiplicative constant $\Omega^2/2$ have been assimilated on the right-hand side of (7).

It is an instructive preliminary to more complete discussion, to consider (7) in simplified and special cases. The neglect of centrifugal terms corresponding to b_1 and c_1 leads to a positive definite function, owing to the circumstance that we have assumed

$$A - B > 0 \quad A - C > 0.$$

An extremum of (7) is verified for values $\alpha = 0$, $\beta = 0$, and (7) is in this case simply the potential energy.

It is in this manner that workers who have adopted directly the Euler equations of rigid body motion, without accounting for relative motion, have concluded that vanishing values of α and β correspond to attitude equilibrium with zero libration. Although one of the Euler equations corresponding to (6) then also suggests the requirement $\gamma = 0$, it is found by directly examining the conditions for extremum that no restriction on γ is in fact indicated: the conditions $\alpha = \beta = 0$ alone assure that the potential energy is stationary.

At this point it should be recalled that the inequalities $A > B > C$ correspond to the stable "spoke" configuration in which the long axis of the satellite is directed toward Earth (the "float" and "arrow" configurations obtained by right angle rotations do not possess the same property of stability); consequently it is sufficient to limit attention to values of α and β less than $\pi/2$. Hence $\alpha = 0$, $\beta = 0$, $\gamma \neq 0$ provides a set of values of the orientation angles for which orientation equilibrium is established, even when centrifugal terms are neglected.

A more accurate representation is considered next for the case of symmetry about Earth-pointing direction: A = B. Then only the second term on the left in equation (7) determines the energy, which is again found to have a stationary value for $\alpha = \beta = 0$, again remaining unrestricted. This "spoke" orientation was identified as an equilibrium orientation by J.L. Synge (Ref. 12), who also set C = 0 to correspond to mass concentrated along the Earth-pointing axis. The isolated equilibrium orientation identified by Synge corresponding to $\alpha = \beta = 0$ is here seen to be generalized for finite inertia moment C: an infinity of values γ also give an extreme value to the Lagrangian function and (7).

In both of the preceding cases it is emphasized that the number of Earth-pointing equilibria is infinite for a finite body (i.e., $C \neq 0$); this fact escapes notice when the Euler equations are examined directly without recognizing that they are redundant for the case of librational equilibrium $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$, as seen from (6) -- also when C = 0.

Before turning to (7) in the general case, useful perspective is obtained by geometrical consideration. In the preceding example in which A = B, equation (7) is seen to represent one hyperbola in a plane with cartesian coordinates c_3, c_1, for each value of the constant; one member of this family of curves corresponds to an extremum of the function. Extremum requirements, expressed by equating to zero the derivatives of (7) with respect to α and β, provide an ellipse in the same plane; the intersection determines unique values of c_3 and c_1, and hence extremizing values of α and β; the arbitrariness of γ provides the infinity of orientations of equilibrium.

It is also useful to recall that in the case of complete dynamical symmetry $A = B = C$, exemplified by a spherical shell, there is neither centrifugal nor gravitational couple, and any spatial orientation is a possible configuration of equilibrium: the number of these can be taken equal to the number of points on the surface of the unit sphere, and thought of as infinity-squared. The body with a single symmetry having been shown to possess a single infinity of equilibria, it is logical to inquire concerning the more general configuration. Equation (7) shows how the number of equilibrium orientations is then determined.

For this purpose it is significant to note that there are in (4) four linearly independent functions b_3, b_1, c_3, c_1, each depending on three variables α, β, γ. This may be regarded as the parametric representation, within a four-dimensional Euclidean manifold, of a sub-set which is a three-dimensional manifold. The extremum requirement isolates within the latter a sub-set which is a two-dimensional manifold, in general, each point of which represents an equilibrium or "permanent" configuration, establishing the fact of a double infinity of such configurations within orientation space of coordinates α, β, γ. This feature was foreshadowed in Ref. 13, and has been verified by direct calculations on high-speed computers of the U.S. Naval Observatory for values of parameters suggested by the lunar mass distribution as it is presently known. A more complete discussion of the geometry of permanent configurations of equilibrium, for elliptic orbits, is intended for later presentation.

Investigation of Effect of Refined Lunar Limb Corrections in the Figure of the Moon

As part of the program for observational re-determination of the constants of the physical libration of the Moon being carried out by the Nautical Almanac Office, U.S. Naval Observatory, it was desired to test the systematic effects on the reduction of introducing Watts' limb corrections as compared to Hayn's limb corrections which were employed by K. Koziel (Ref. 16).

To make this test the Dorpat heliometer observations of 1884 and 1885 were chosen. These observations have been completely discussed and rigorously reduced by K. Koziel. The reduced observations, as presented there, were modified by the introduction of Watts' limb corrections (Ref. 17) and a new solution made. As a control on the process of reforming the conditional equations and normal equations, a solution without limb corrections was also made.

The results of the solution without limb corrections is shown at the top of Figure 2; λ, β and h represent the selenographic longitude and latitude of Mösting A and its radius vector, i.e., its distance from the center of the Moon at the mean distance of the Moon from the Earth. I is the angle between the Moon's equator and the ecliptic, $f = \alpha/\beta$ is the mechanical ellipticity of the Moon, and R is the radius of the Moon. The insignificant differences from the same solution made by Koziel arise from assigning unit weight to all the observational equations, and to small differences in decimal accuracy of the computations.

The results of the solution using Watts' limb corrections is shown in the lower left of Figure 2. For comparison, the results of the previous solution using Hayn's limb corrections is shown to the right.

The mean error of unit weight of the three solutions is shown in the upper right portion of the figure. While the error using Watts' limb corrections is less than when using Hayn's limb corrections, there appear to be residual errors in the observations themselves that are larger than the limb effects.

The systematic shift in latitude introduced by Watts in forming his definitive datum surface is reflected in the latitude of Mösting A resulting from the solution using his limb corrections. A slightly different value of the inclination of the Moon's equator to the ecliptic also is apparent.

For lack of sufficient observational weight, no discussion of the derived value of f is given, nor has any attempt been made to solve for the coefficients of the free libration in longitude.

Acknowledgment

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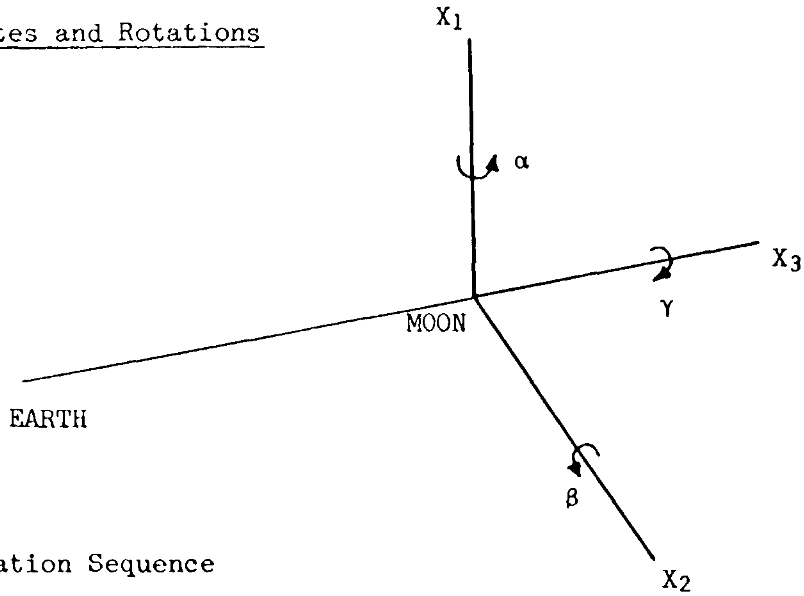
Coordinates and Rotations



Ordered Rotation Sequence

i) about X_1-axis
ii) about (displaced) X_2-axis
iii) about (displaced) X_3-axis

DIRECTION COSINES, GENERAL ROTATION

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$a_1 = \cos\beta\cos\gamma$

$b_1 = -\cos\beta\sin\gamma$

$c_1 = \sin\beta$

$a_3 = \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma$

$b_3 = \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma$

$c_3 = \cos\alpha\,\mathrm{oos}\beta$

Fig. 1. Reference Axes, Rotational and Angle Conventions, and Axis Transformations

Solution Without Limb Corrections

λ	-5° 11' 56"	± 15"	p.e.	Mean Error of Unit Weight	
β	-3° 10' 29"	± 16"			
h	15' 32".49	± ".57		No. L.C.	±.8313
I	1° 31' 9"	± 22"		Watts	±.5302
f	0.70	± .06		Hayn	-.6091
R	15' 32".93	± ".03			

Solution With Limb Corrections

	Watts	p.e.	Hayn
λ	-5° 12' 14"	±10"	-5° 11' 50"
β	-3° 11' 24"	±10"	-3° 10' 27"
h	15' 32".52	±".37	15' 32".88
I	1° 31' 53"	±14"	1° 31' 36"
f	0.65	±.04	0.71
R	15' 32".87	±".02	15' 32".88

Fig. 2. Solutions for Lunar Constants, Effects of Limb Corrections

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